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STRONGLY INDEFINITE FUNCTIONALS AND MULTIPLE SOLUTIONS OF ELLIPTIC SYSTEMS

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ABSTRACT. We study existence and multiplicity of solutions of the elliptic system

$$\begin{cases} -\Delta u = H_u(x,u,v) & \text{in } \Omega, \\ -\Delta v = -H_v(x,u,v) & \text{in } \Omega, \quad u(x) = v(x) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 3$, is a smooth bounded domain and $H \in \mathcal{C}^1(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$. We assume that the nonlinear term

$$H(x,u,v) \sim |u|^p + |v|^q + R(x,u,v) \quad \text{with} \quad \lim_{|(u,v)| \to \infty} \frac{R(x,u,v)}{|u|^p + |v|^q} = 0,$$

where $p \in (1,\ 2^*)$, $2^* := 2N/(N-2)$, and $q \in (1,\ \infty)$. So some supercritical systems are included. Nontrivial solutions are obtained. When $H(x,u,v)$ is even in (u,v), we show that the system possesses a sequence of solutions associated with a sequence of positive energies (resp. negative energies) going toward infinity (resp. zero) if $p > 2$ (resp. $p < 2$). All results are proved using variational methods. Some new critical point theorems for strongly indefinite functionals are proved.

1. Introduction and main results

Consider the following elliptic system:

$$\text{(E)} \qquad \begin{cases} -\Delta u = H_u(x,u,v) & \text{in } \Omega, \\ -\Delta v = -H_v(x,u,v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 3$, is a smooth bounded domain and $H : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^1$-function. Here H_u denotes the partial derivative of H with respect to the variable u. Writing $z := (u,v)$, we suppose $H(x,0) \equiv 0$ and $H_z(x,0) \equiv 0$. Then $z = 0$ is a trivial solution of the system. In this paper we discuss the existence of nontrivial solutions. Roughly speaking, we are mainly interested in the class of Hamiltonians H such that

$$H(x,u,v) \sim |u|^p + |v|^q + R(x,u,v) \quad \text{with} \quad \lim_{|z| \to \infty} \frac{R(x,u,v)}{|u|^p + |v|^q} = 0,$$

where $1 < p < 2^* := 2N/(N-2)$ and $q > 1$. The most interesting results obtained here refer to the case when $q \geq 2^*$, which correspond to critical and supercritical problems. The case when $q < 2^*$ has been studied by Costa and Magalhães [5],

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[6] and Benci and Rabinowitz [3]. See also Bartsch and De Figueiredo [2], De Figueiredo and Magalhães [7], De Figueiredo and Felmer [8] and Hulshof and van der Vorst [11], where similar systems also leading to strongly indefinite functionals have been studied. However, only subcritical systems have been considered in those papers.

Letting $2_* = 2^*/(2^*-1) = 2N/(N+2)$, we assume that $H(x,z)$ satisfies the following condition:

(H_0) there are $p \in (1,\, 2^*)$, $q \in (1, \infty)$ and $\tau \in (1,\ 1+q/2_*)$ such that, for all (x,z),
$$|H_u(x,u,v)| \le \gamma_0(1+|u|^{p-1}+|v|^{\tau-1})$$
and
$$|H_v(x,u,v)| \le \gamma_0(1+|u|^{p-1}+|v|^{q-1}).$$

In all hypotheses on $H(x,z)$ the γ_i's denote positive constants independent of (x,z). We note that if $q<2^*$, then $2_* < q/(q-1)$, i.e., $q-1<q/2_*$. Hence, it is possible that $q \le \tau < 1+q/2_*$. However, if $q \ge 2^*$, then $\tau < q$. Furthermore, we remark that τ can be very large, if q is sufficiently large.

In addition, we need distinct conditions on H corresponding to the cases when $p>2$, $p<2$ or $p=2$.

First, consider the case when $p>2$. In this case, we assume the following three conditions:

(H_1) there are $\mu>2$, $\nu>1$ and $R_1 \ge 0$ such that
$$\frac{1}{\mu}H_u(x,z)u + \frac{1}{\nu}H_v(x,z)v \ge H(x,z) \quad \text{whenever } |z| \ge R_1\,,$$
with the provision that $\nu=\mu$ if $q>2$;

(H_2) there are $2_*(p-1) \le \alpha \le p$ and $2_*(\tau-1) < \beta$ such that
$$H(x,z) \ge \gamma_1\left(|u|^\alpha + |v|^\beta\right) - \gamma_2 \quad \text{for all } (x,z),$$
and $\beta = q$ if $q > 2^*$;

(H_3) $H(x,0,v) \ge 0$ and $H_u(x,u,0) = o(|u|)$ as $u \to 0$ uniformly in x.

We prove the following results.

Theorem 1.1. *Let* (H_0) *be satisfied with* $p>2$. *If* $(H_1)-(H_3)$ *hold, then* (E) *has at least one nontrivial solution.*

In order to provide some more transparent hypotheses under which the above result holds, we next present some conditions on H that are sufficient for (H_0), (H_1) and (H_2) to hold:

(H_0') there are $p \in (1,\, 2^*)$ and $q \in (2, \infty)$ such that, for all (x,z),
$$|H_u(x,u,v)| \le \gamma_0(1+|u|^{p-1}+|v|^{\frac{q}{2}-1})$$
and
$$|H_v(x,u,v)| \le \gamma_0(1+|u|^{p-1}+|v|^{q-1});$$

(H_1') there are $\mu>2$ and $R_1 \ge 0$ such that
$$H_u(x,z)u + H_v(x,z)v \ge \mu H(x,z) \quad \text{whenever } |z| \ge R_1\,;$$

(H_2') for p and q as above,
$$H(x,z) \ge \gamma_1\left(|u|^p + |v|^q\right) - \gamma_2 \quad \text{for all } (x,z).$$

Theorem 1.1′. *Let* (H_0') *be satisfied with* $p > 2$. *If* $(H_1'), (H_2')$, *and* (H_3) *hold, then* (E) *has at least one nontrivial solution.*

Theorem 1.2. *Let* (H_0) *be satisfied with* $p > 2$. *If* $H(x,z)$ *is even in* z *and satisfies* (H_1) *and* (H_2), *then* (E) *has a sequence* (z_n) *of solutions with energies* $I(z_n) := \int_\Omega \left(\frac{1}{2}(|\nabla u_n|^2 - |\nabla v_n|^2) - H(x,z_n)\right)$, *going to* ∞ *as* $n \to \infty$.

In order to describe the other results, let $\sigma(-\Delta)$ denote the set of all eigenvalues of $(-\Delta, H_0^1(\Omega))$: $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$.

We now consider the case when $p < 2$. We make the following assumptions:

(H_4) there are $\mu \in (1,\ 2)$, $\nu \ge 2$ and $\gamma_3 \ge 0$ ($\gamma_3 = 0$, if $q > 2^*$) such that
$$H(x,u,v) \ge \frac{1}{\mu} H_u(x,u,v)u + \frac{1}{\nu} H_v(x,u,v)v - \gamma_3 \quad \text{for all } (x,z);$$

(H_5) there are $\alpha \in (1,\ 2)$ and $\delta \in (0,\ 1/2)$ such that $H(x,u,v) \ge \gamma_4 |u|^\alpha - \delta\lambda_1 v^2$ for all (x,z);

(H_6) if $q \ge 2^*$, then $H_v(x,z)v \ge \gamma_5 |v|^q - \gamma_6(|v| + u^2)$ for all (x,z).

With these assumptions we have the following three results, for the case when $p < 2$.

Theorem 1.3. *Suppose that* (H_0) *holds with* $p < 2$ *and* $q \ge 2$. *If* $H(x,z)$ *also satisfies* $(H_4) - (H_6)$, *then* (E) *has at least one nontrivial solution.*

Theorem 1.4. *Suppose that* $H(x,z)$ *is even in* z *and* (H_0) *holds with* $p < 2$ *and* $q \ge 2$. *If* $H(x,z)$ *also satisfies* $(H_4)-(H_6)$, *then* (E) *has a sequence* (z_n) *of solutions with negative energies* $I(z_n)$ *going to* 0 *as* $n \to \infty$.

Theorem 1.5. *Let* (H_0), *with* $p, q \in (1,\ 2)$, *and* (H_5) *be satisfied. Then* (E) *has at least one nontrivial solution. If, in addition,* $H(x,z)$ *is even in* z, *then* (E) *has a sequence* (z_n) *of solutions with negative energies* $I(z_n)$ *going to* 0 *as* $n \to \infty$.

Finally, we consider the case when $p = 2$, which presents some sort of resonance. Assume

(H_7) there exist $b_0 \le 0 < a_0$ such that $R_0(x,z) := H(x,z) - \frac{1}{2}(a_0 u^2 + b_0 v^2) = o(|z|^2)$ as $z \to 0$ uniformly in x;

(H_8) there exist $\sigma \in (1,\ 2)$, $a_\infty \in [a_0,\ \infty)\backslash\sigma(-\Delta)$, such that $R_\infty(x,z) := H(x,z) - \frac{1}{2}a_\infty u^2$ satisfies $|\partial_u R_\infty(x,z)| \le \gamma_7(1 + |u|^{\sigma-1} + |v|^{\tau-1})$ and $R_\infty(x,z) \ge \gamma_8 |v|^q - \gamma_9\,(1 + |u|^\sigma)$.

The position of the numbers a_0, a_∞, b_0 with respect to the spectrum $\sigma(-\Delta)$ plays a very essential role in the next result. For that matter, let i, j, k be nonnegative integers such that $\lambda_i = \min\{\lambda \in \sigma(-\Delta) : \ \lambda > a_0\}$, $\lambda_j = \max\{\lambda \in \sigma(-\Delta) : \ \lambda < -b_0\}$, $\lambda_k = \max\{\lambda \in \sigma(-\Delta) : \ \lambda < a_\infty\}$, and set
$$\ell = \begin{cases} j & \text{if } a_\infty = a_0, \\ j + k - i + 1 & \text{if } a_\infty > a_0. \end{cases}$$
Now we can state our last result.

Theorem 1.6. *Let* (H_0) *be satisfied with* $p = 2$ *and* $\tau < 1 + q/2$. *Assume that* $H(x,z)$ *is even in* z *and satisfies* (H_7) *and* (H_8). *Then* (E) *has at least one pair of nontrivial solutions if* $\ell = 1$, *and infinitely many solutions if* $\ell \ge 2$.

The cases covered in Theorem 1.6 include some asymptotically linear systems. Such systems have been studied in [5], [6] and Silva [13]. However, their results are not comparable with the ones obtained here.

We organize the paper as follows. In order to establish multiplicity of solutions we need some new abstract propositions on critical point theory for strongly indefinite functionals, which will be provided in Section 2. These propositions are based on certain Galerkin approximations, and we emphasize that the functionals do not satisfy the usual Palais-Smale condition. In Section 3 we study systems that are superlinear in the variable u, and prove Theorems 1.1 and 1.2. In Section 4 we consider systems that are sublinear in the variable u, and prove Theorems 1.3, 1.4 and 1.5. In both Sections 3 and 4, the variable v can have subcritical growth as well as supercritical growth. Finally, in Section 5, we consider a special asymptotically linear system and prove existence of multiple solutions.

2. Critical points for strongly indefinite functionals

Let E be a Banach space with norm $\|\cdot\|$. Suppose that E has a direct sum decomposition $E = E^1 \oplus E^2$ with both E^1 and E^2 being infinite dimensional. Let P^1 denote the projection from E onto E^1. Assume (e^1_n) (resp. (e^2_n)) is a basis for E^1 (resp. E^2). Set

$$X_n := \operatorname{span}\{e^1_1, \cdots, e^1_n\} \oplus E^2, \quad X^m := E^1 \oplus \operatorname{span}\{e^2_1, \cdots, e^2_m\},$$

and let $(X^m)^\perp$ denote the complement of X^m in E. For a functional $I \in \mathcal{C}^1(E, \mathbb{R})$ we set $I_n := I|_{X_n}$, the restriction of I on X_n. Recall that a sequence $(z_j) \subset E$ is said to be a $(\mathrm{PS})^*_c$ sequence if $z_j \in X_{n_j},\ n_j \to \infty,\ I(z_j) \to c$ and $I'_{n_j}(z_j) \to 0$ as $j \to \infty$. If any $(\mathrm{PS})^*_c$ sequence has a convergent subsequence, then we say that I satisfies the $(\mathrm{PS})^*_c$ condition.

Denote the upper and lower level sets, respectively, by $I_a = \{z \in E : I(z) \geq a\}$, $I^b = \{z \in E : I(z) \leq b\}$ and $I^b_a = I_a \cap I^b$ (denote similarly $(I_n)_a$, $(I_n)^b$ and $(I_n)^b_a$). We also set $\mathcal{K} = \{z \in E : I'(z) = 0\}$, $\mathcal{K}_c = \mathcal{K} \cap I_c$, $\mathcal{K}^c = \mathcal{K} \cap I^c$ and $\mathcal{K}^b_a = \mathcal{K}_a \cap \mathcal{K}^b$.

Proposition 2.1. *Let E be as above and let $I \in \mathcal{C}^1(E, \mathbb{R})$ be even with $I(0) = 0$. In addition, suppose that, for each $m \in \mathbb{N}$, the conditions below hold:*

- (I_1) *there is $R_m > 0$ such that $I(z) \leq 0$ for all $z \in X^m$ with $\|z\| \geq R_m$;*
- (I_2) *there are $r_m > 0$ and $a_m \to \infty$ such that $I(z) \geq a_m$ for all $z \in (X^{m-1})^\perp$ with $\|z\| = r_m$;*
- (I_3) *I is bounded from above on bounded sets of X^m;*
- (I_4) *if $c \geq 0$, any $(\mathrm{PS})^*_c$ sequence (z_n) has a subsequence along which $z_n \rightharpoonup z \in \mathcal{K}_c$.*

Then the functional I has a sequence (c_k) of critical values, with the property that $c_k \to \infty$.

Remark 2.1. This proposition is more or less known if the condition (I_4) is replaced by the $(\mathrm{PS})^*$ condition (cf. [1], [9]), or by the usual Palais-Smale condition, that is, *any sequence $(z_k) \subset E$ such that $|I(z_k)| \leq c$ and $I'(z_k) \to 0$ has a convergent subsequence* (cf. [3]).

Proposition 2.2. *Let E be as above and let $I \in \mathcal{C}^1(E, \mathbb{R})$ be even. Assume that $I(0) = 0$ and that, for each $m \in \mathbb{N}$, the two conditions below hold:*

- (I_5) *there are $r_m > 0$ and $a_m > 0$ such that $I(z) \geq a_m$ for all $z \in X^m$ with $\|z\| = r_m$;*
- (I_6) *there is $b_m > 0$ with $b_m \to 0$ such that $I(z) \leq b_m$ for all $z \in (X^{m-1})^\perp$.*

Moreover, suppose that either I satisfies the $(\mathrm{PS})_c^*$ *condition for all* $c > 0$, *or that the condition below holds:*

- (I_7) $\inf I(\mathcal{K}) = 0$, *and, for all* $c \geq 0$, *any* $(\mathrm{PS})_c^*$ *sequence* (z_n) *has a subsequence along which* $z_n \rightharpoonup z \in \mathcal{K}^c$ *with* $z = 0$ *only if* $c = 0$.

Then I has a sequence (c_k) of positive critical values satisfying $c_k \to 0$.

Proof. Let Σ be the family of symmetric, closed subsets of $E\backslash\{0\}$, and let $\gamma : \Sigma \to \mathbb{N} \cup \{0, \infty\}$ denote the Krasnoselski genus map. Set

$$c_n^m := \sup_{A \in \Sigma_n^m} \inf_{z \in A} I(z),$$

where

$$\Sigma_n^m := \{A \in \Sigma : \ A \subset X_n \text{ and } \gamma(A) \geq n + m\}.$$

Fix $m \in \mathbb{N}$. The Borsuk-Ulam theorem implies that $A \cap (X^{m-1})^\perp \neq \emptyset$ for each $A \in \Sigma_n^m$. It follows from (I_6) that

$$\inf_{z \in A} I(z) \leq \sup_{z \in (X^{m-1})^\perp} I(z) \leq b_m.$$

On the other hand, since $\gamma(\partial B_{r_m} \cap X_n^m) = n+m$, one has $S_n^m := \partial B_{r_m} \cap X_n^m \in \Sigma_n^m$, and so, by (I_5), we obtain

$$\inf_{z \in S_n^m} I(z) \geq a_m.$$

Therefore,

$$a_m \leq c_n^m \leq b_m. \tag{2.1}$$

A standard deformation argument, using a positive pseudo-gradient flow, yields the existence of a sequence $(z_n^m)_{n=1}^\infty$, with $z_n^m \in X_n$ satisfying

$$|I(z_n^m) - c_n^m| \leq \frac{1}{n} \quad \text{and} \quad \|I_n'(z_n^m)\| \leq \frac{1}{n}.$$

We can assume that $I(z_n^m) \to c_m$ as $n \to \infty$. So, (z_n^m) is a $(\mathrm{PS})_{c_m}^*$ sequence with

$$a_m \leq c_m \leq b_m. \tag{2.2}$$

Now, if we assume that I satisfies the $(\mathrm{PS})_c^*$ condition for $c > 0$, then the conclusion follows. Next, suppose instead that (I_7) holds. Then, along a subsequence, $z_n^m \rightharpoonup z_m$ as $n \to \infty$ with $I'(z_m) = 0$ and $0 < I(z_m) \leq c_m$. Finally, by (2.2),

$$I(z_m) \leq b_m \to 0,$$

and the proof is complete. □

Proposition 2.3. *Let E be as above and let $I \in \mathcal{C}^1(E, \mathbb{R})$ be even with $I(0) = 0$. Suppose, in addition, that the three conditions below hold:*

- (I_8) *there are* $\ell \in \mathbb{N}$ *and* r, $a > 0$ *such that* $I(z) \geq a$ *for all* $z \in X^\ell$ *with* $\|z\| = r$;
- (I_9) *there is* $b > 0$ *such that* $\sup I(E^2) \leq b$;
- (I_{10}) *any* $(\mathrm{PS})_c^*$, $c > 0$, *sequence* (z_n) *has a subsequence along which* $z_n \rightharpoonup z \in \mathcal{K}_c$ *and* $P^1 z_n \to P^1 z$.

Then I has at least one pair of nontrivial critical points if $\ell = 1$, and infinitely many critical points if $\ell > 1$, with positive critical values.

Proof. Let $\Sigma,\ \gamma,\ \Sigma_n^m$ and c_n^m be as in the proof of Proposition 2.2. As before, by (I_8) and (I_9), we obtain

$$a \le c_n^m \le b \quad \text{for all } n \in \mathbb{N} \text{ and } m = 1, \cdots, \ell,$$

and we find sequences $z_n^m \in X_n$ such that, going to subsequences if necessary, $I(z_n^m) \to c_m$ and $I_n'(z_n^m) \to 0$ as $n \to \infty$ with

$$b \ge c_1 \ge c_2 \ge \cdots \ge c_\ell \ge a.$$

Using (I_{10}), we can assume furthermore that $z_n^m \rightharpoonup z_m \in \mathcal{K}_{c_m}$ for $m = 1, \cdots, \ell$, as $n \to \infty$. If $\ell = 1$ the proof is complete.

Consider $\ell > 1$. Let $F = \{z \in \mathcal{K} :\ I(z) > 0\}$. We are going to prove that F is an infinite set. Arguing by contradiction, we suppose that F is finite. Choose $0 < \mu < a \le b < \nu$ satisfying

$$\mu < \inf I(F) \le \sup I(F) < \nu.$$

Let $k \in \mathbb{N}$ be so large that $0 \notin A := Q^k F$, where $Q^k : E \to X^k$ denotes the projection. Then A is also finite, and $\gamma(A) = 1$. By the continuity of γ, for all $\delta > 0$ small, $\gamma(N_\delta^k(A)) = \gamma(A)$, where $N_\delta^k(A) = \{z \in X^k :\ \operatorname{dist}(z, A) \le \delta\}$. Set $C_\delta = N_\delta^k(A) \oplus (X^k)^\perp$. Since $N_\delta^k(A) \subset C_\delta$ and $Q^k : C_\delta \to N_\delta^k(A)$, it follows from the properties of γ that $\gamma(C_\delta) = \gamma(N_\delta^k(A))$. We remark that $Q^k = P^1 + (Q^k - P^1)$ and that the range of $Q^k - P^1$ is k-dimensional. So by virtue of (I_{10}), we conclude that, for all $c \ge 0$, any $(\mathrm{PS})_c^*$ sequence (z_n) has a subsequence along which $z_n \rightharpoonup z \in \mathcal{K}_c$ and $Q^k z_n \to Q^k z$. Hence there are $n_0 \in \mathbb{N}$ and $\sigma > 0$ such that for all $n \ge n_0$,

$$\|I_n'(w)\| \ge \sigma \quad \text{for all } w \in (I_n)_\mu^\nu \setminus C_\delta^n,$$

where $C_\delta^n = C_\delta \cap X_n$. By a standard deformation argument, we can then construct a sequence of odd homeomorphisms $\eta_n : X_n \to X_n$ such that

$$\eta_n\left((I_n)_\mu \setminus C_\delta^n\right) \subset (I_n)_\nu$$

(cf. [12]). For n_0 sufficiently large, we can suppose that

$$\mu < c_n^\ell \le c_n^{\ell-1} \le \cdots \le c_n^1 < \nu \quad \text{for all } n \ge n_0.$$

Let $G \in \Sigma_n^\ell$ be such that $\inf I(G) > (\mu + c_n^\ell)/2$. One then has

$$\eta_n\left(G \backslash C_\delta^n\right) \subset (I_n)_\nu$$

and

$$\begin{aligned}\gamma(\eta_n(G\backslash C_\delta^n)) = \gamma(G\backslash C_\delta^n) &\ge \gamma(G) - \gamma(C_\delta^n) \\ &\ge n + \ell - \gamma(C_\delta^n) \ge n + \ell - 1.\end{aligned}$$

Thus $\eta_n(G\backslash C_\delta^n) \in \Sigma_n^{\ell-1}$ and $\nu \le \inf I(\eta_n(G\backslash C_\delta^n)) \le c_n^{\ell-1}$. One finally comes to $\nu \le c_n^{\ell-1} < \nu$, which is a contradiction. □

From now on we turn to the system (E). We denote by $|\cdot|_t$ the usual $L^t(\Omega)$ norm for all $t \in [1,\ \infty]$. For $q > 1$ let $V_q = H_0^1(\Omega)$ if $q \le 2^*$ and $V_q = H_0^1(\Omega) \cap L^q(\Omega)$, the Banach space equipped with the norm $\|v\|_{V_q} = \left(|\nabla v|_2^2 + |v|_q^2\right)^{1/2}$, if $q > 2^*$. Let E_q be the product space $H_0^1(\Omega) \times V_q$ with elements denoted by $z = (u, v)$. We denote the norm in E_q by $\|z\|_q = (|\nabla u|_2^2 + \|v\|_{V_q}^2)^{1/2}$. E_q has the direct sum decomposition

$$E_q = E_q^- \oplus E^+, \quad z = z^- + z^+$$

where

$$E_q^- = \{0\} \times V_q \quad \text{and} \quad E^+ = H_0^1(\Omega) \times \{0\}.$$

For convenience, we will write $z^+ = u$ and $z^- = v$. Recall that by $(\lambda_n)_{n\in\mathbb{N}}$ we denote the sequence of eigenvalues of $(-\Delta, H_0^1(\Omega))$. Let e_n, $|e_n|_2 = 1$, be the eigenfunction corresponding to λ_n for each $n \in \mathbb{N}$. Clearly, $e_n^+ := (e_n, 0)$, $n \in \mathbb{N}$, is a basis for E^+, and $e_n^- = (0, e_n), n \in \mathbb{N}$, is a basis for E_q^-.

Suppose that the assumption (H_0) holds. Then

$$H(x,z) \le c(1 + |u|^{2^*} + |v|^q) \quad \text{for all } (x,z). \tag{2.3}$$

So the functional

$$I(z) := \frac{1}{2}\int_\Omega (|\nabla u|^2 - |\nabla v|^2) - \int_\Omega H(x,z) \tag{2.4}$$

is well defined in E_q. Moreover, $I \in \mathcal{C}^1(E_q, \mathbb{R})$, and the critical points of I are the solutions of (E).

Lemma 2.1. *If (H_0) holds, then I' is weakly sequentially continuous, that is, $I'(z_n) \rightharpoonup I'(z)$ provided $z_n \rightharpoonup z$.*

Proof. If $q < 2^*$ this statement is well known. Assume now that $q \ge 2^*$. Let $z_n \rightharpoonup z$ in E_q. Clearly, for all $w = (\varphi, \psi) \in E_q$, we have

$$\int_\Omega (\nabla u_n \nabla\varphi - \nabla v_n \nabla\psi) \to \int_\Omega (\nabla u \nabla\varphi - \nabla v \nabla\psi)\,.$$

So it remains to show that

$$\int_\Omega H_u(x,z_n)\varphi \;\to\; \int_\Omega H_u(x,z)\varphi \quad \text{for all } \varphi \in H_0^1(\Omega) \tag{2.5}$$

and

$$\int_\Omega H_v(x,z_n)\psi \;\to\; \int_\Omega H_v(x,z)\psi \quad \text{for all } \psi \in V_q. \tag{2.6}$$

By the Sobolev embedding theorem and using interpolation, we obtain that $u_n \to u$ in L^t for $t \in [1,\ 2^*)$ and $v_n \to v$ in L^t for $t \in [1,\ q)$. Noting that $|H_u(x,u,v)| \le \gamma_0(1 + |u|^{p-1} + |v|^{\tau-1})$ with $2_*(\tau - 1) < q$, (2.5) follows easily since $u_n \to u$ in L^p, $v_n \to v$ in $L^{2_*(\tau-1)}$ and $\varphi \in H_0^1(\Omega) \subset L^{2^*}$. Next we see that (2.6) is clearly true when $\psi \in L^\infty$. In general, for a $\psi \in V_q$ we proceed as follows. Let $\tilde\psi_m \in L^\infty$ with $\tilde\psi_m \to \psi$ in L^q as $m \to \infty$. So

$$\left|\int_\Omega (H_v(x,z_n) - H_v(x,z))\psi\right| = \left|\int_\Omega (H_v(x,z_n) - H_v(x,z))(\tilde\psi_m + (\psi - \tilde\psi_m))\right|,$$

and using (H_0) we see that this expression is less than the following sum:

$$\begin{aligned}&\left|\int_\Omega (H_v(x,z_n) - H_v(x,z))\tilde\psi_m\right| \\ &\quad + c_1\left(|\psi - \tilde\psi_m|_1 + |u_n|_p^{p-1}|\tilde\psi_m - \psi|_p + |v_n|_q^{q-1}|\tilde\psi_m - \psi|_q\right),\end{aligned}$$

which by its turn is estimated by

$$\left|\int_\Omega (H_v(x,z_n) - H_v(x,z))\tilde\psi_m\right| + c_2\left(|\tilde\psi_m - \psi|_p + |\tilde\psi_m - \psi|_q\right),$$

since (z_n) is bounded in E_q and L^∞ is dense in L^q. So (2.6) is proved, and it follows that

$$I'(z_n)w \to I'(z)w \text{ for all } w \in E_q.$$

$\square$

3. The case $p > 2$

Throughout this section let (H_0) be satisfied with $p > 2$, and assume that (H_1) and (H_2) hold. Observe that, by (H_2), there exists $R > 0$ such that $H(x,z) > 0$ whenever $|z| \geq R$. This, jointly with (H_1), implies

$$H(x,z) \geq c_1(|u|^\mu + |v|^\nu) - c_2 \quad \text{for all } (x,z) \tag{3.1}$$

(see [10]). This, together with (2.3) and (H_2), shows that

$$\nu \leq q \ \text{ and } \ \beta \leq q. \tag{3.2}$$

Moreover, by virtue of (3.1) and (H_2), we may assume, without loss of generality, that (since $\mu > 2$)

$$\alpha > 2\,. \tag{3.3}$$

Now we set $E^1 = E_q^-$, $E^2 = E^+$ and $e_n^1 = e_n^-$, $e_n^2 = e_n^+$ for all $n \in \mathbb{N}$. So $E_q = E_1 \oplus E_2$. Consider the functional defined by (2.4), which has the properties stated in Section 2.

Lemma 3.1. *Any* $(\mathrm{PS})_c^*$ *sequence is bounded.*

Proof. Let $z_n \in X_n$ be such that

$$I(z_n) \to c \quad \text{and} \quad I_n'(z_n) \to 0.$$

Case 1: $q \leq 2$. In this case $E_q = (H_0^1(\Omega))^2$. By (H_1), for $w_n := (\frac{1}{\mu}u_n, \frac{1}{\nu}v_n)$, we have

$$\begin{aligned} &I(z_n) - I_n'(z_n)w_n \\ &= (\frac{1}{2} - \frac{1}{\mu})|\nabla u_n|_2^2 + (\frac{1}{\nu} - \frac{1}{2})|\nabla v_n|_2^2 \\ &\quad + \int_\Omega \left(\frac{1}{\mu}H_u(x,z_n)u_n + \frac{1}{\nu}H_v(x,z_n)v_n - H(x,z_n)\right) - c_1 \\ &\geq (\frac{1}{2} - \frac{1}{\mu})|\nabla u_n|_2^2 + (\frac{1}{\nu} - \frac{1}{2})|\nabla v_n|_2^2 - c_2. \end{aligned} \tag{3.4}$$

If $q < 2$, then (3.2) shows that $\nu < 2$, and so $\|z_n\|_q^2 \leq c_3(1 + \|z_n\|_q)$, which implies that (z_n) is bounded in E_q. Assume $q = 2$. Invoking (3.2), we get $\nu \leq 2$, and so $|\nabla u_n|_2^2 \leq c(1 + \|z_n\|_q)$ by (3.4). Since $H(x,z) > 0$ for all $|z|$ large, and

$$\frac{1}{2}|\nabla v_n|_2^2 + \int_\Omega H(x,z_n) = -I(z_n) + \frac{1}{2}|\nabla u_n|_2^2 \leq c(1 + \|z_n\|_q),$$

one sees that $\|z_n\|_q^2 \leq c(1 + \|z_n\|_q)$. Hence, (z_n) is bounded.

Case 2: $q > 2$. Note that in this case $\nu = \mu > 2$ in (H_1). So

$$\begin{aligned} I(z_n) - \frac{1}{2}I_n'(z_n)z_n &= \int_\Omega (\frac{1}{2}H_z(x,z_n)z_n - H(x,z_n)) \\ &\geq (\frac{\mu}{2} - 1)\int_\Omega H(x,z_n) - c, \end{aligned} \tag{3.5}$$

which, together with (H_2), yields

$$|u_n|_\alpha^\alpha + |v_n|_\beta^\beta \leq c(1 + \|z_n\|_q). \tag{3.6}$$

Using (H_0), we get

$$
\begin{aligned}
|\nabla u_n|_2^2 &= I_n'(z_n)(u_n,0) + \int_\Omega H_u(x,z_n)u_n \\
&\le c_1\|z_n\|_q + c_2\int_\Omega (|u_n|^p + |v_n|^{\tau-1}|u_n|).
\end{aligned} \tag{3.7}
$$

Next we estimate the integrals in the right side of (3.7). Since $2_*(p-1) \le \alpha \le p$, we have that $\theta := \alpha/(1+\alpha-p) \le 2^*$. Using the Hölder inequality, the Sobolev embedding theorem and (3.6), we obtain

$$
\int_\Omega |u_n|^p \le |u_n|_\alpha^{p-1}|u_n|_\theta \le c_1 + c_2\|z_n\|_q^{1+(p-1)/\alpha}.
$$

Similarly, since $\tau - 1 < \beta/2_*$, we have $1 < \omega := \beta/(1+\beta-\tau) < 2^*$, and hence

$$
\int_\Omega |v_n|^{\tau-1}|u_n| \le |v_n|_\beta^{\tau-1}|u_n|_\omega \le c_1 + c_2\|z_n\|_q^{1+(\tau-1)/\beta}.
$$

Therefore, using the estimate in (3.7), we obtain

$$
|\nabla u_n|_2^2 \le c(1 + \|z_n\|_q^{1+(p-1)/\alpha} + \|z_n\|_q^{1+(\tau-1)/\beta}). \tag{3.8}
$$

Since

$$
|\nabla v_n|_2^2 = -I_n'(z_n)(0,v_n) - \int_\Omega H_z(x,z_n)z_n + \int_\Omega H_u(x,z_n)u_n,
$$

and using (3.5) and the above arguments, we obtain

$$
|\nabla v_n|_2^2 \le c(1 + \|z_n\|_q^{1+(p-1)/\alpha} + \|z_n\|_q^{1+(\tau-1)/\beta}). \tag{3.9}
$$

Recall that, in view of our assumptions, $(p-1)/\alpha \le 1/2_*$, $(\tau-1)/\beta < 1/2_*$, and $\beta = q$ if $q > 2^*$. Hence, it follows from (3.6) and (3.8)-(3.9) that (z_n) is bounded in E_q. □

Lemma 3.2. *Let $z_n \in X_n$ be a $(\mathrm{PS})_c^*$ sequence. If $q \le 2^*$, then (z_n) contains a convergent subsequence. If $q > 2^*$, then there is a $z \in E_q$ such that, along a subsequence, $z_n \rightharpoonup z$ and $I'(z) = 0$ and $I(z) \ge c$.*

Proof. By Lemma 3.1, (z_n) is bounded. We can assume that $z_n \rightharpoonup z$ in E_q, $z_n \to z$ in $(L^s(\Omega))^2$ for all $1 \le s < 2^*$, and $z_n(x) \to z(x)$ a.e. on Ω. It follows from the weak sequential continuity of I' (see Lemma 2.1) that $I'(z) = 0$. Since $I_n'(z_n) \to 0$, we obtain

$$
\begin{aligned}
(\nabla u_n, \nabla u_n - \nabla u)_{L^2} &= I_n'(z_n)(u_n - u, 0) + \int_\Omega H_u(x,z_n)(u_n-u) \\
&= o(1) + \int_\Omega H_u(x,z_n)(u_n-u).
\end{aligned}
$$

Using (H_0) and the Hölder inequality, we obtain the estimate

$$
\begin{aligned}
&\left|\int_\Omega H_u(x,z_n)(u_n-u)\right| \\
&\le c\Big(|u_n-u|_1 + |u_n|_p^{p-1}|u_n-u|_p + |v_n|_\beta^{\tau-1}|u_n-u|_\omega\Big) = o(1),
\end{aligned}
$$

where ω is as in the proof of Lemma 3.1. Hence $|\nabla u_n|_2^2 \to |\nabla u|_2^2$, which implies $u_n \to u$ in $H_0^1(\Omega)$. Let $P_n : E_q \to X_n$ denote the projection. Observe that $P_n z \to z$

in E_q for all $z \in E_q$. Moreover, using again (H_0) and the Hölder inequality, we estimate

$$\begin{aligned}&\left|\int_\Omega H_v(x,z_n)(v-P_nv)\right|\\ &\leq c\left(|v-P_nv|_1+|u_n|_p^{p-1}|v-P_nv|_p+|v_n|_q^{q-1}|v-P_nv|_q\right)\to 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\nabla v_n,\nabla v-\nabla v_n)_{L^2}&=o(1)+I_n'(z_n)(0,v_n-P_nv)+\int_\Omega H_v(x,z_n)(v_n-P_nv)\\ &=o(1)+\int_\Omega H_v(x,z_n)(v_n-v)\\ &=o(1)+\int_\Omega H_z(x,z_n)(z_n-z)-\int_\Omega H_u(x,z_n)(u_n-u)\\ &=o(1)+\int_\Omega H_z(x,z_n)z_n-\int_\Omega H_z(x,z_n)z.\end{aligned}$$

Lebesgue's theorem and the weak sequential continuity of $H_z(x,\cdot)$ (see the proof of Lemma 2.1) yield

$$|\nabla v|_2^2-\limsup_{n\to\infty}|\nabla v_n|_2^2=\liminf_{n\to\infty}\left(\int_\Omega H_z(x,z_n)z_n-\int_\Omega H_z(x,z_n)z\right)\geq 0,$$

i.e., $|\nabla v|_2^2\geq\limsup_{n\to\infty}|\nabla v_n|_2^2$. This, together with the weak lower semicontinuity of norms, implies $|\nabla v_n|_2\to|\nabla v|_2$. So $v_n\to v$ in $H_0^1(\Omega)$.

Therefore, if $q\leq 2^*$, we obtain that, along a subsequence, $z_n\to z$ in E_q and consequently $I(z)=c$. Next assume that $q>2^*$. Observe that

$$\begin{aligned}I(z)-I(z_n)=&\frac{1}{2}(|\nabla u|_2^2-|\nabla u_n|_2^2)-\frac{1}{2}(|\nabla v|_2^2-|\nabla v_n|_2^2)\\ &+\int_\Omega H(x,z_n)-\int_\Omega H(x,z);\end{aligned}$$

hence,

$$I(z)-c=o(1)+\int_\Omega H(x,z_n)-\int_\Omega H(x,z).$$

Lebesgue's theorem then yields

$$I(z)-c=\liminf_{n\to\infty}\int_\Omega H(x,z_n)-\int_\Omega H(x,z)\geq 0,$$

that is, $I(z)\geq c$. □

Lemma 3.3. *If (H_3) also holds, there are $r,\rho>0$ such that* $\inf I(\partial B_rE^+)\geq\rho$.

Proof. By (H_0) and (H_3), for any $\varepsilon>0$, there is $c_\varepsilon>0$ such that

$$H(x,u,0)\leq\varepsilon|u|^2+c_\varepsilon|u|^{2^*}.$$

Hence,

$$I(u)\geq\frac{1}{2}|\nabla u|_2^2-\varepsilon|u|_2^2-c_\varepsilon|u|_{2^*}^{2^*},$$

and the conclusion follows easily. □

Let $e\in E^+$ with $|\nabla e|_2^2=1$, and set

$$Q=\{(se,v):\ 0\leq s\leq r_1,\|v\|_q\leq r_2\}.$$

Lemma 3.4. *If* (H_3) *also holds, there are* $r_1, r_2 > 0$, *with* $r_1 > r$, *such that* $I(z) \le 0$ *for all* $z \in \partial Q$.

Proof. By (H_3), $I(z) \le 0$ for all $z \in E_q^-$. By (H_2),

$$I((se, v)) \le \frac{s^2}{2} - \frac{1}{2}|\nabla v|_2^2 - c_1 \int_\Omega (|se|^\alpha + |v|^\beta) + c_2.$$

The conclusion follows since $\alpha > 2$. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Lemmas 3.3 and 3.4 say that I has the linking geometry. Let $Q_n := Q \cap X_n$, and define

$$c_n := \inf_{\gamma \in \Gamma_n} \max I(\gamma(Q_n)),$$

where $\Gamma_n := \{\gamma \in \mathcal{C}(Q_n, X_n) : \ \gamma|_{\partial Q_n} = id\}$. Then $\rho \le c_n \le \kappa := \sup I(Q)$. A standard deformation argument shows that there is $z_n \in X_n$ such that $|I(z_n) - c_n| \le 1/n$ and $\|I_n'(z_n)\| \le 1/n$. So we obtain a $(\mathrm{PS})_c^*$ sequence (z_n) with $c \in [\rho, \kappa]$. Lemma 3.2 implies $z_n \rightharpoonup z$ with $I'(z) = 0$ and $I(z) \ge c$. The proof is complete. □

We now consider the multiplicity of solutions using Proposition 2.1.

Lemma 3.5. *I satisfies* (I_1).

Proof. Using (H_2), we obtain

$$I(z) \le \frac{1}{2}|\nabla u|_2^2 - \frac{1}{2}|\nabla v|_2^2 - c_1 \int_\Omega \left(|u|^\alpha + |v|^\beta\right) + c_2.$$

Since all norms in $\mathrm{span}\{e_1, \cdots, e_m\}$ are equivalent, we obtain

$$I(z) \le -\Big(c_3|\nabla u|_2^{\alpha-2} - \frac{1}{2}\Big)|\nabla u|_2^2 - \Big(\frac{1}{2}|\nabla v|_2^2 + c_1|v|_\beta^\beta\Big) + c_2,$$

for all $z = (u, v) \in X^m \simeq \mathrm{span}\{e_1, \cdots, e_m\} \times V_q$. So (I_1) follows easily. □

Lemma 3.6. *I satisfies* (I_2).

Proof. Since $(X^m)^\perp \subset H_0^1(\Omega)$ and $H_0^1(\Omega)$ embeds compactly in $L^p(\Omega)$, we have that $\eta_m > 0$ and $\eta_m \to 0$ as $m \to \infty$, where

$$\eta_m := \sup_{u \in (X^m)^\perp \setminus \{0\}} \frac{|u|_p}{|\nabla u|_2}; \tag{3.10}$$

see Lemma 3.8 in [14]. For $z = (u, 0) \in (X^m)^\perp$, it follows from (H_0) that

$$\begin{aligned} I(z) &= \frac{1}{2}|\nabla u|_2^2 - \int_\Omega H(x, u, 0) \ge \frac{1}{2}|\nabla u|_2^2 - c_1|u|_p^p - c_2 \\ &\ge \frac{1}{2}|\nabla u|_2^2 - c_1\eta_m^p|\nabla u|_2^p - c_2. \end{aligned}$$

Setting $r_m = (pc_1\eta_m^p)^{1/(2-p)}$ and $a_m = (p-2)r_m^2/2p - c_2$, we come to the desired conclusion. □

Proof of Theorem 1.2. Since $H(x, z)$ is even in z, I is even. Lemma 3.2 shows that I satisfies the assumption (I_4) of Proposition 2.1. Lemmas 3.5 and 3.6 show that (I_1) and (I_2) hold. Clearly (I_3) is also true. Therefore by Proposition 2.1, there is a sequence $(z_n) \subset E_q$ satisfying $I'(z_n) = 0$ and $I(z_n) \to \infty$. The proof is complete. □

4. The case $p < 2$

Throughout this section we assume that (H_0) is satisfied with $p \in (1,\ 2)$. We also suppose that $(H_4) - (H_6)$ hold.

Let $E_q = E^1 \oplus E^2$ be as in Section 3. Consider the functional

$$J(z) = -I(z) = \int_\Omega H(x,z) + \frac{1}{2}|\nabla v|_2^2 - \frac{1}{2}|\nabla u|_2^2.$$

Lemma 4.1. *Any* $(\mathrm{PS})_c^*$ *sequence* (z_n) *has a subsequence converging weakly to a critical point* z *of* J *with* $J(z) \le c$*, and* $z = 0$ *only if* $z_n \to 0$ *in* E_q.

Proof. The proof is divided into two parts.

Part I. The sequence (z_n) is bounded in E_q. By (H_4) it follows that

$$J(z_n) - J_n'(z_n)(\frac{1}{\mu}u_n, \frac{1}{\nu}v_n) \ge \Big(\frac{1}{2} - \frac{1}{\nu}\Big)|\nabla v_n|_2^2 + \Big(\frac{1}{\mu} - \frac{1}{2}\Big)|\nabla u_n|_2^2 - c.$$

Hence $|\nabla u_n|_2^2 \le c(1 + \|z_n\|_q)$. If $\nu > 2$, we also get $|\nabla v_n|_2^2 \le c(1 + \|z_n\|_q)$. If $\nu = 2$, we use (H_5) and the fact that $|\nabla v|_2^2 \ge \lambda_1 |v|_2^2$ in order to obtain

$$\Big(\frac{1}{2} - \delta\Big)|\nabla v_n|_2^2 \le \frac{1}{2}|\nabla v_n|_2^2 + \int_\Omega H(x,z_n) = J(z_n) + \frac{1}{2}|\nabla u_n|_2^2.$$

Hence, $|\nabla v_n|_2^2 \le c(1 + \|z_n\|_q)$, and we get

$$|\nabla u_n|_2^2 + |\nabla v_n|_2^2 \le c(1 + \|z_n\|_q).$$

Thus, if $q \le 2^*$, then (z_n) is bounded in E_q. Assume next that $q > 2^*$. It follows from (H_6) that

$$J_n'(z_n)(0, v_n) \ge c_1|v_n|_q^q + |\nabla v_n|_2^2 - c_2\left(|v_n|_1 + |u_n|_2^2\right). \tag{4.1}$$

Thus $|\nabla u_n|_2^2 + |\nabla v_n|_2^2 + |v_n|_q^q \le c(1 + \|z_n\|_q)$, which implies that (z_n) is bounded in E_q also in the case when $q > 2^*$.

Part II. We can now suppose that $z_n \rightharpoonup z$ in E_q, $z_n \to z$ in $(L^s(\Omega))^2$ for all $1 \le s < 2^*$, and $z_n(x) \to z(x)$ a.e. in $x \in \Omega$. It follows that z is a critical point of J. As in the proof of Lemma 3.2, using (H_0) and

$$J_n'(z_n)(u_n - u, 0) = \int_\Omega H_u(x,z_n)(u_n - u) - (\nabla u_n, \nabla(u_n - u))_{L^2},$$

we obtain that

$$\begin{aligned}&|(\nabla u_n, \nabla(u_n - u))_{L^2}|\\ &\le o(1) + c(|u_n - u|_1 + |u_n|_p^{p-1}|u_n - u|_p + |v_n|_\beta^{\tau-1}|u_n - u|_\omega) = o(1),\end{aligned}$$

and so $u_n \to u$ in $H_0^1(\Omega)$. Let $P_n : E_q \to X_n$ be the projection as in the proof of Lemma 3.2. So we obtain

$$\begin{aligned}(\nabla v_n, \nabla(v - v_n))_{L^2} &= o(1) + (\nabla v_n, \nabla(P_n v - v_n))_{L^2}\\ &= o(1) + \int_\Omega H_v(x,z_n)(v_n - P_n v) - J_n'(z_n)(0, v_n - P_n v)\\ &= o(1) + \int_\Omega H_v(x,z_n)(v_n - v) + \int_\Omega H_v(x,z_n)(v - P_n v).\end{aligned}$$

Using (H_0), we have

$$\left|\int_\Omega H_v(x,z_n)(v - P_n v)\right| \le c\left(1 + |u_n|_{2^*}^{2^*} + |v_n|_q^{q-1}\right)\|v - P_n v\|_q \to 0.$$

Consequently,

$$(\nabla v_n, \nabla(v - v_n))_{L^2} = \int_\Omega H_v(x, z_n)(v_n - v) + o(1). \tag{4.2}$$

Thus if $q < 2^*$, it follows from (4.2) that $|\nabla v_n|_2 \to |\nabla v|_2$, which implies $v_n \to v$, and so $z_n \to z$. This proves that J satisfies the $(\mathrm{PS})_c^*$ condition in this case, and that $J(z) = c$.

Consider next $q \geq 2^*$. The weak sequential continuity of $H_v(x, \cdot)$ (see the proof of Lemma 2.1) yields $\int_\Omega H_v(x, z_n)v \to \int_\Omega H_v(x, z)v$. By (H_6), $f_n(x) := H_v(x, z_n)v_n + \gamma_6(|v_n| + |u_n|^2) \geq 0$. Using the fact that $|v_n|_1 \to |v|_1$ and $|u_n|_2 \to |u|_2$, and applying Fatou's lemma to the sequence (f_n), we get

$$\liminf_{n\to\infty} \int_\Omega H_v(x, z_n)v_n \geq \int_\Omega H_v(x, z)v.$$

Using this estimate in (4.2), we obtain that $|\nabla v|_2^2 \geq \limsup_{n\to\infty} |\nabla v_n|_2^2$, which implies that $v_n \to v$ in $H_0^1(\Omega)$. In order to conclude that $J(z) \leq c$, we use the estimate

$$J(z_n) - J(z) = \int_\Omega (H(x, z_n) - H(x, z)) + o(1),$$

(H_4) and Fatou's lemma. Finally, if $z = 0$, then $z_n \to 0$ in $(H_0^1(\Omega))^2$. By (4.1),

$$|v_n|_q^q \leq o(1) + c\left(|v_n|_1 + |u_n|_2^2\right) \to 0,$$

and so $z_n \to 0$. □

Remark 4.1. In a similar way, using even simpler arguments, one checks that, if (H_0) holds with $p, q \in (1,\ 2)$, J satisfies the $(\mathrm{PS})_c^*$ condition for all c.

Remark 4.2. Let $\tilde{J}_m = J|_{X^m}$ denote the restriction of J on X^m. As in Lemma 4.1, it is not difficult to check that, if the sequence $(z_m) \subset E_q$, with $z_m \in X^m$, satisfies $J(z_m) \to c$ and $\tilde{J}'_m(z_m) \to 0$ as $m \to \infty$, then it possesses a subsequence converging weakly to a critical point z of J with $J(z) \leq c$, and $z = 0$ only if $z_n \to 0$ in E_q. We also have, as in Remark 4.1, that, if (H_0) holds with $p, q \in (1,\ 2)$, then any such sequence has a convergent subsequence.

Lemma 4.2. *There is an $R > 0$ such that $J(z) \leq 0$ for all $z = (u, 0)$ with $\|z\| \geq R$.*

Proof. By (H_0), we have $H(x, u, 0) \leq c(1 + |u|^p)$. Hence

$$\begin{aligned} J((u, 0)) &= \int_\Omega H(x, u, 0) - \frac{1}{2}|\nabla u|_2^2 \leq c_1 + c_2|u|_p^p - \frac{1}{2}|\nabla u|_2^2 \\ &\leq c_1 - \left(\frac{1}{2}|\nabla u|_2^{2-p} - c_3\right)|\nabla u|_2^p, \end{aligned}$$

and the lemma follows, since $p < 2$. □

Lemma 4.3. *For $\varepsilon > 0$ small there is $\rho > 0$ such that $J((\varepsilon e_1, v)) \geq \rho$ for all $v \in V_q$, where e_1 is the eigenfunction corresponding to the first eigenvalue λ_1 of $(-\Delta,\ H_0^1(\Omega))$.*

Proof. By (H_5), for $\varepsilon > 0$ small, $H(x, \varepsilon e_1, v) \geq \gamma_4 \varepsilon^\alpha e_1^\alpha - \delta\lambda_1 v^2$; hence,

$$J((\varepsilon e_1, v)) = \int_\Omega H(x, \varepsilon e_1, v) + \frac{1}{2}|\nabla v|_2^2 - \frac{1}{2}\lambda_1\varepsilon^2 \geq (\gamma_4|e_1|_\alpha^\alpha - \frac{1}{2}\lambda_1\varepsilon^{2-\alpha})\varepsilon^\alpha.$$

The conclusion follows. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Recall that $X^m \simeq \text{span}\{e_1, \cdots, e_m\} \times V_q$, and consider the restrictions $\tilde{J}_m$ as defined in Remark 4.2. Set $D_R = B_R \cap E^2 = B_R \cap (H_0^1(\Omega) \times \{0\})$ and $D_m = D_R \cap X^m$, where $R > 0$ comes from Lemma 4.2. Define

$$c_m := \inf_{\gamma \in \Gamma_m} \max J(\gamma(D_m)),$$

where $\Gamma_m := \{\gamma \in \mathcal{C}(D_m, X^m) : \ \gamma(z) = z \text{ for all } z \in \partial D_m\}$. It is well known that $\gamma(D_m) \cap W \neq \emptyset$ for all $\gamma \in \Gamma_m$, where $W = \{(\varepsilon e_1, 0)\} \times V_q$ with $\varepsilon > 0$ small. Invoking Lemma 4.3, we fix an $\varepsilon > 0$ so small that there is $\rho > 0$ satisfying $\inf J(W) \geq \rho$. Then we have

$$\rho \leq c_m \leq b := \max J(D_R).$$

The well-known saddle point theorem (cf. [12] or [4], [14]) implies that there is $z_m \in X^m$ satisfying $|J(z_m) - c_m| \leq 1/m$ and $\|\tilde{J}_m'(z_m)\| \leq 1/m$. Now by virtue of Remark 4.2, along a subsequence, $z_m \rightharpoonup z$ with $J'(z) = 0$ and $z \neq 0$, ending the proof. □

We now turn to the proof of Theorems 1.4 and 1.5.

Lemma 4.4. *If, in addition,* $\gamma_3 = 0$ *in* (H_4)*, then* J *satisfies* (I_5)*.*

Proof. It follows from (H_5) that

$$\begin{aligned} J(z) &\geq c_1|u|_\alpha^\alpha + \Big(\frac{1}{2} - \delta\Big)|\nabla v|_2^2 - \frac{1}{2}|\nabla u|_2^2 \\ &\geq \Big(c_2 - \frac{1}{2}|\nabla u|^{2-\alpha}\Big)|\nabla u|_2^\alpha + \Big(\frac{1}{2} - \delta\Big)|\nabla v|_2^2. \end{aligned} \tag{4.3}$$

Since $\alpha < 2$, the result follows in the case when $q \leq 2^*$. Next consider $q > 2^*$. Suppose (I_5) does not hold. Then for any $r > 0$ there is a sequence $z_j \in X^m$ such that $\|z_j\| = r$ and $J(z_j) \to 0$. It follows from (4.3) with $z = z_j$, and for r small, that $|\nabla u_j|_2 \to 0$ and $|\nabla v_j|_2 \to 0$. All this implies that $\int_\Omega H(x, z_j) \to 0$. From assumption (H_0) and the fact that (u_j) lies in a finite-dimensional subspace, it follows that $\int_\Omega H_u(x, z_j)u_j \to 0$. Consequently, by (H_4) with $\gamma_3 = 0$, $\int_\Omega H_v(x, z_j)v_j \to 0$. This, jointly with (H_6), yields

$$|v_j|_q^q \leq c_1 \int_\Omega H_v(x, z_j)v_j + c_2(|v_j|_1 + |u_j|_2^2) \ \to \ 0.$$

Hence, $z_j \to 0$ in E_q, which is a contradiction. □

Lemma 4.5. J *satisfies* (I_6)*.*

Proof. By (H_0), $H(x, u, 0) \leq c(|u| + |u|^p)$, and so, for $u \in (X^{m-1})^\perp$, one has

$$\begin{aligned} J((u, 0)) &\leq c_1\left(|u|_p + |u|_p^p\right) - \frac{1}{2}|\nabla u|_2^2 \\ &\leq \Big(c_1|u|_p - \frac{1}{4}|\nabla u|_2^2\Big) + \Big(c_1|u|_p^p - \frac{1}{4}|\nabla u|_2^2\Big) \\ &\leq \Big(c_1\eta_m - \frac{1}{4}|\nabla u|_2\Big)|\nabla u|_2 + \Big(c_1\eta_m^p - \frac{1}{4}|\nabla u|_2^{2-p}\Big)|\nabla u|_2^p, \end{aligned}$$

where η_m was defined by (3.10). Let $b_m := (c_1\eta_m)^2 + (1-p/2)c_1\eta_m^p(2pc_1\eta_m^p)^{p/(2-p)}$. Then $0 < b_m \to 0$ and $J((u, 0)) \leq b_m$ for all $(u, 0) \in (X^{m-1})^\perp$. □

Proof of Theorem 1.4. Since $H(x,z)$ is even in z, J is even. If $q \le 2^*$, then J satisfies the $(\mathrm{PS})_c^*$ condition for all c (see the proof of Lemma 4.1). If $q > 2^*$, then, using assumption (H_4) applied to a critical point z, we obtain

$$J(z) = J(z) - J_n'(z)(\frac{1}{\mu}u, \frac{1}{\nu}v) \ge \Big(\frac{1}{2} - \frac{1}{\nu}\Big)|\nabla v|_2^2 + \Big(\frac{1}{\mu} - \frac{1}{2}\Big)|\nabla u|_2^2 \ge 0.$$

This, jointly with Lemma 4.1, shows that (I_7) is satisfied. It follows from Lemmas 4.4 and 4.5 that J satisfies (I_5) and (I_6). Therefore, the desired conclusion follows. $\square$

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. The proof of the existence of one nontrivial solution is similar to that of Theorem 1.3, using Remark 4.2 and Lemmas 4.2 and 4.3. The other conclusion can be obtained along the lines of the proof of Theorem 1.4, using Remark 4.1 and Lemmas 4.4 and 4.5. $\square$

5. The case $p = 2$

In this section we always assume that (H_0) holds with $p = 2$ and $\tau < 1 + q/2$. We also suppose that (H_7) and (H_8) are satisfied. We will apply Proposition 2.3 in order to prove Theorem 1.6. Thus, set

$$E^2 = \mathrm{span}\{e_1^+, \cdots, e_k^+\} \oplus E_q^- \simeq \mathrm{span}\{e_1, \cdots, e_k\} \times V_q\,, \qquad E^1 = E_q \ominus E^2,$$

and

$$X^\ell = E^1 \oplus \mathrm{span}\{e_i^+, \cdots, e_k^+, e_1^-, \cdots, e_j^-\}.$$

One may arrange the bases as $e_n^1 = e_{k+n}^+$ for $n \in \mathbb{N}$, and $e_n^2 = e_{n+i-1}^+$ for $1 \le n \le \ell - j$, $e_n^2 = e_{n-\ell+j}^-$ for $\ell - j < n \le \ell$, $e_n^2 = e_{n-\ell}^+$ for $\ell < n \le \ell + i - 1$, and $e_n^2 = e_{n-k}^-$ for $n > \ell + i - 1$. Consider the functional I given by (2.4).

Lemma 5.1. *I satisfies (I_8); that is, there exist $r, a > 0$ such that $I(z) \ge a$ for all $z \in X^\ell$ with $\|z\|_q = r$.*

Proof. Let $z = (u,v) \in X^\ell$. Since $v \in \mathrm{span}\{e_1, \cdots, e_j\}$, we have $v \in L^\infty$. By (H_0) and (H_7), for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$R_0(x,z) \le \varepsilon |z|^2 + c_\varepsilon(|u|^{2^*} + |v|^q).$$

Thus

$$\begin{aligned} I(z) &= \frac{1}{2}\left(|\nabla u|_2^2 - a_0|u|_2^2\right) - \frac{1}{2}\left(|\nabla v|_2^2 - b_0|v|_2^2\right) - \int_\Omega R_0(x,z) \\ &\ge \frac{1}{2}\Big(1 - \frac{a_0}{\lambda_i}\Big)|\nabla u|_2^2 + \frac{1}{2}\Big(\frac{-b_0}{\lambda_j} - 1\Big)|\nabla v|_2^2 - \varepsilon|z|_2^2 - c_\varepsilon\left(|u|_{2^*}^{2^*} + |v|_q^q\right). \end{aligned}$$

Now the conclusion follows easily. $\square$

Lemma 5.2. *I satisfies (I_9); that is,* $\sup I(E^2) < \infty$.

Proof. For $z \in E^2$ we have, using (H_8), that

$$\begin{aligned} I(z) &= \frac{1}{2}\left(|\nabla u|_2^2 - a_\infty |u|_2^2\right) - \frac{1}{2}|\nabla v|_2^2 - \int_\Omega R_\infty(x,z) \\ &\le -\frac{1}{2}\Big(\frac{a_\infty}{\lambda_k} - 1\Big)|\nabla u|_2^2 - \frac{1}{2}|\nabla v|_2^2 + \gamma_9 |u|_\sigma^\sigma - \gamma_8 |v|_q^q + \gamma_9|\Omega| \\ &\le -\Big(\frac{1}{2}\Big(\frac{a_\infty}{\lambda_k} - 1\Big)|\nabla u|^{2-\sigma} - c_1\Big)|\nabla u|_2^\sigma - \Big(\frac{1}{2}|\nabla v|_2^2 + \gamma_8 |v|_q^q\Big) + c_2, \end{aligned}$$

which implies that $I(z) \le 0$ for all $z \in E^2$ with $\|z\|_q$ large. □

Lemma 5.3. *Let $c > 0$. Then any* $(\mathrm{PS})_c$ *sequence is bounded.*

Proof. We decompose $H_0^1(\Omega)$ as

$$H_0^1(\Omega) = U^- \oplus U^+, \quad u = u^- + u^+,$$

where $U^- = \mathrm{span}\{e_1, \cdots, e_k\}$ and U^+ is the orthogonal complement of U^- in $H_0^1(\Omega)$.

Let (z_n) be a $(\mathrm{PS})_c^*$ sequence. Using the expression of I_n':

$$I_n'(z_n)u_n^+ = |\nabla u_n^+|_2^2 - a_\infty |u_n^+|_2^2 - \int_\Omega \partial_u R_\infty(x, z_n)u_n^+\,,$$

plus (H_8) and the Hölder inequality, we obtain

$$\Big(1 - \frac{a_\infty}{\lambda_{k+1}}\Big)|\nabla u_n^+|_2^2 \le c_1 |\nabla u_n^+|_2 + \gamma_7\Big(|u_n^+|_1 + |u_n|_\sigma^{\sigma-1}|u_n^+|_\sigma + |v_n|_q^{\tau-1}|u_n^+|_r\Big)$$

where $r = q/(1 + q - \tau)$. By assumption, $1 < r < 2$. It then follows from the Sobolev embedding theorems that

$$\Big(1 - \frac{a_\infty}{\lambda_{k+1}}\Big)|\nabla u_n^+|_2^2 \le c_2\Big(1 + |u_n|_\sigma^{\sigma-1} + |v_n|_q^{\tau-1}\Big)|\nabla u_n^+|_2.$$

Similarly, we deduce that

$$\Big(\frac{a_\infty}{\lambda_k} - 1\Big)|\nabla u_n^-|_2^2 \le c_2\Big(1 + |u_n|_\sigma^{\sigma-1} + |v_n|_q^{\tau-1}\Big)|\nabla u_n^-|_2.$$

The two previous inequalities imply the estimate

$$|\nabla u_n|_2^2 \le c_3\left(1 + |u_n|_\sigma^{2(\sigma-1)} + |v_n|_q^{2(\tau-1)}\right). \tag{5.1}$$

Using the expression of H given in (H_8), and recalling that $I(z_n) > 0$ for large n, we obtain

$$\frac{1}{2}|\nabla v_n|_2^2 + \int_\Omega R_\infty(x, z_n) = \frac{1}{2}|\nabla u_n|_2^2 - \frac{a_\infty}{2}|u_n|_2^2 - I(z_n) \le \frac{1}{2}|\nabla u_n|_2^2. \tag{5.2}$$

Next using (5.2), assumption (H_8) and (5.1), we obtain

$$|\nabla v_n|_2^2 + c_4 |v_n|_q^q \le c_5\left(1 + |u_n|_\sigma^\sigma + |u_n|_\sigma^{2(\sigma-1)} + |v_n|_q^{2(\tau-1)}\right). \tag{5.3}$$

The combination of (5.1) and (5.3) implies

$$|\nabla z_n|_2^2 + |v_n|_q^q \le c_6\left(1 + |u_n|_2^\sigma + |v_n|_q^{2(\tau-1)}\right).$$

Since $\sigma < 2$ and $2(\tau - 1) < q$, we see that (z_n) is bounded. □

Lemma 5.4. *I satisfies* (I_{10}).

Proof. Let (z_n) be a $(\mathrm{PS})_c^*$ sequence with $c > 0$. Using Lemma 5.3, an argument similar to that of Lemma 3.2 shows that along a subsequence $z_n \rightharpoonup z \in \mathcal{K}_c$, we have $u_n \to u$ in $H_0^1(\Omega)$. Since $E^1 \subset H_0^1(\Omega)$, we have $P^1 z_n \to P^1 z$. □

Proof of Theorem 1.6. Since $H(x, z)$ is even in z, I is even. By assumption, $I(0) = 0$. Lemmas 5.1, 5.2 and 5.4 show that I satisfies $(I_8) - (I_{10})$. Now Proposition 2.3 applies, and the proof is complete. □

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